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1993 J. Phys. A: Math. Gen. 26 2409

(http://iopscience.iop.org/0305-4470/26/10/012)

J. Phys. A: Math. Gen. 26 (1993) 2409–2419. Printed in the UK

Sum rules for zeros of Bessel functions and an application to spherical Aharonov–Bohm quantum bags

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Received 21 December 1992

Abstract. We study the sum $\zeta_H(s) = \sum_j E_j^{-s}$ over the eigenvalues E_j of the Schrödinger equation in a spherical domain with Dirichlet walls, threaded by a line of magnetic flux. Rather than using Green function techniques, we tackle the mathematically non-trivial problem of finding exact sum rules for the zeros of Bessel functions J_ν, which are extremely helpful when seeking numerical approximations to ground state energies. These results are particularly valuable if ν is neither an integer nor half an odd integer.

Dedicated to David Bohm, *in memoriam.*

1. Introduction

There is a large class of physically interesting problems where the Hamiltonian is closely related to the operator of Bessel's equation, this includes, e.g.: (i) classical equations for vibrating strings and drumheads, heat conduction in cylinders, normal modes in resonant cavities, Fraunhofer difraction through circular apertures, etc.; (ii) quantum particles which move freely within cylindrically or spherically symmetric domains, on whose boundaries the wave function vanishes (they play the role of reflecting or bouncing walls). The two-dimensional case has been further complicated in two different ways: by altering the shape of the boundary—quantum billiards [1, 2]—and by introducing magnetic potentials such that the domain is threaded by a line of magnetic flux—Aharonov–Bohm quantum billiards [2].

This paper deals with the extension of the last situation to spherical domains of arbitrary radius a, diametrically threaded by a flux line. For such a system, the time-independent Schrödinger equation takes the form

$$\frac{1}{2\mu}[-i\hbar\nabla - qA(r)]^2\Psi(r) = E\Psi(r) \tag{1.1}$$

where μ is the mass of the particle, q its charge and $A(r)$ is some suitable vector potential, which produces a straight line of flux along the z axis. To this end we choose

$$A(r) = \frac{\Phi}{2\pi}\frac{1}{r\sin\theta}e_\varphi \tag{1.2}$$

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where Φ is the value of the flux and e_φ denotes the unitary tangent vector for increasing φ (the usual spherical coordinates, r, θ and φ, are employed). This corresponds to selecting a gauge in which the lines of the vector potential become concentric circles. After noticing that $\nabla \cdot A = 0$, as should be expected, (1.1) reduces to

$$\left[\nabla^2 + k^2 - \frac{1}{r^2 \sin^2\theta}\left(2\mathrm{i}\alpha\frac{\partial}{\partial\varphi} + \alpha^2\right)\right]\Psi(r,\theta,\varphi) = 0 \tag{1.3}$$

where $k^2 = 2\mu E/\hbar^2$ and $\alpha = (q/\hbar)\Phi/2\pi$. Applying the usual variable separation method, we set $\Psi(r,\theta,\varphi) = R(r)\mathcal{Y}(\theta,\varphi)$ and arrive at

$$\left[\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + k^2r^2 - C\right]R(r) = 0 \tag{1.4}$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2} - 2\mathrm{i}\alpha\frac{\partial}{\partial\varphi} - \alpha^2\right) + C\right]\mathcal{Y}(\theta,\varphi) = 0 \tag{1.5}$$

with C a separation constant. Writing $C = \beta(\beta+1)$, (1.4) becomes the spherical Bessel equation of parameter β, not necessarily integer. Thus, the solution regular at $r = 0$ is

$$R(r) \sim j_\beta(kr) = \sqrt{\frac{\pi}{2kr}}J_{\beta+1/2}(kr) \tag{1.6}$$

where j_β denotes the spherical Bessel function. As a result of the boundary condition imposed on the surface $r = a$, ka must be a zero of $J_{\beta+1/2}$. The energy is therefore discretized as

$$E_{\beta n} = \frac{\hbar^2}{2\mu a^2}j^2_{\beta+1/2,n} \tag{1.7}$$

$j_{\beta+1/2,n}$ being the nth positive zero of $J_{\beta+1/2}$.

Taking $\mathcal{Y}(\theta,\varphi) = \Theta(\theta)\phi(\varphi)$ with the function $\phi(\varphi)$ of the form $\mathrm{e}^{\mathrm{i}m\varphi}$, $m \in \mathbb{Z}$, and performing the change of variables $x = \cos\theta$, equation (1.5) yields

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}}{\mathrm{d}x} + \beta(\beta+1) - \frac{(m-\alpha)^2}{1-x^2}\right]\Theta(\theta(x)) = 0 \tag{1.8}$$

and writing $\Theta(\theta(x)) = (1-x^2)^{|m-\alpha|/2}u(x)$, the function $u(x)$ satisfies the following equation

$$(1-x^2)\ u'' - 2x\ (|m-\alpha|+1)\ u' + [C - |m-\alpha|(|m-\alpha|+1)]\ u = 0. \tag{1.9}$$

Since $x = 0$ is a regular point, we make the expansion $u(x) = \sum_{k=0}^{\infty} a_k x^k$, which leads to the recurrence relation

$$a_{k+2} = \frac{(k+|m-\alpha|)(k+|m-\alpha|+1) - C}{(k+2)(k+1)}a_k. \tag{1.10}$$

Good behaviour at $x = \pm 1$ can only be achieved by truncating the series into a polynomial. This takes place if there is some $p \in \mathbb{N}$ such that $C = (|m-\alpha|+p)(|m-\alpha|+p+1)$. This recurrence relation would be the same as the one for Legendre's associated functions

P_l^m, were it not for the presence of a (generally) *non-integer* α. Therefore, we adopt the notation

$$\Theta(\theta) \sim P_\beta^{m-\alpha}(\cos\theta) \qquad |m-\alpha| \leqslant \beta \tag{1.11}$$

where $\beta \equiv |m-\alpha| + p$ is, as a rule, also non-integer. The general form of the wave function is a superposition of the type

$$\Psi(r,\theta,\varphi) = \sum_{n,l,m} c_{nlm}\, j_\beta\left(j_{\beta+1/2,n}\frac{r}{a}\right) P_\beta^{m-\alpha}(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\varphi}. \tag{1.12}$$

The angular eigenfunctions $\sim P_\beta^{m-\alpha}(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\varphi}$ generalize the spherical harmonics $Y_{lm}(\theta,\varphi)$, corresponding to $\alpha = 0$. It is precisely for this reason—i.e. the presence of α—that the solutions are no longer eigenfunctions of the angular momentum $\boldsymbol{L}^2$ (they are instead eigenfuctions of the mechanical momentum Π^2). This was to be expected, as radial symmetry has been broken by $\boldsymbol{A}(\boldsymbol{r})$. The energy spectrum has now become

$$E_{\beta,n} = \frac{\hbar^2}{2\mu a^2} j^2_{\beta+1/2,n} \qquad m \in \mathbb{Z} \qquad \beta = |m-\alpha| + p \qquad p \in \mathbb{N}. \tag{1.13}$$

The ground state energy is therefore $E_{0,1} = (\hbar^2/2\mu a^2) j^2_{|\alpha|+1/2,1}$, for $|\alpha| \leqslant 1/2$. Notice that $\alpha = 1/2$ gives rise to double degeneracy, since states with m and $-m+1$ have the same energy. This is also true, in particular, for the ground state.

Let us recall that the zeta function of a Hermitean operator A with infinitely many discrete eigenvalues λ_i is defined as

$$\zeta_A(s) = \sum_i \frac{1}{\lambda_i^s} = \mathrm{Tr} A^{-s}. \tag{1.14}$$

An alternative to the computer evaluation of these Bessel function zeros is their approximation by means of sum rules. When A has a discrete spectrum $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, several ways to find estimates of low-lying eigenvalues—quoted in [4]—are possible. The most suitable here is Euler's method (originally applied to the zeros of J_0), which is based on the inequalities

$$\zeta_A(s)^{-1/s} < \lambda_1 < \frac{\zeta_A(s)}{\zeta_A(s+1)}. \tag{1.15}$$

In fact, $\lim_{s\to\infty}[\zeta_A(s)]^{-1/s} = \lim_{s\to\infty}(\zeta_A(s)/\zeta_A(s+1)) = \lambda_1$ and, prior to taking the limit, (1.15) always holds.

In our case, we will take A to be the operator of the spherical Bessel equation (1.4) with $C \equiv (|m-\alpha|+p)(|m-\alpha|+p+1)$. Thus, for each possible l, the zeta function $\zeta_A(s)$ to be considered is of the type $\zeta_A(s) \sim \zeta_\nu(2s)$, with

$$\zeta_\nu(s) = \sum_{n=1}^{\infty} \frac{1}{j_{\nu n}^s} \qquad \mathrm{Re}\ s > 1 \tag{1.16}$$

where $j_{\nu n}$ is the nth non-vanishing zero of J_ν. The condition $\mathrm{Re}\ s > 1$ stems from considering the asymptotic form of the $j_{\nu n}$'s, which is roughly $(\nu - \frac{1}{2})\pi/2 + \pi n$, and from comparing with the Hurwitz zeta function. At any rate, the existence of poles at some values of s ought to be checked by operator heat-kernel expansion methods.

2. Recursive rules for zeta functions associated with Bessel operators

2.1. Quadratic recursion

When written in the form of an infinite product involving its non-vanishing zeros, the νth Bessel function of the first species reads

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu n}^2}\right). \tag{2.1}$$

Differentiating the log of this expression with respect to z, and using the identity

$$J_\nu'(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z) \tag{2.2}$$

we arrive at

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = 2z \sum_{n=1}^{\infty} \frac{1}{j_{\nu n}^2 - z^2} \tag{2.3}$$

and, since $j_{\nu 1} < j_{\nu 2} < \cdots$, taking $|z| < j_{\nu 1}$ we are able to put

$$\frac{1}{1 - (z/j_{\nu n})^2} = \sum_{m=0}^{\infty} \left(\frac{z}{j_{\nu n}}\right)^{2m}.$$

Then, interchanging the summations and looking at (1.16) we arrive at

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = 2 \sum_{n=1}^{\infty} \zeta_\nu(2n) z^{2n-1}. \tag{2.4}$$

This formula will constitute our starting point and is a well known result (see e.g. [3, p 61]). Now one can, of course, find the values of the zeta functions through

$$\zeta_\nu(2n) = \frac{1}{2(2n-1)!} \frac{\mathrm{d}^{2n-1}}{\mathrm{d}z^{2n-1}} \frac{J_{\nu+1}(z)}{J_\nu(z)} \bigg|_{z=0}.$$

Rather than doing this, we take the derivative of (2.4) with respect to z, and making use of (2.2) and

$$J_{\nu+1}'(z) = J_\nu(z) - \frac{\nu+1}{z} J_{\nu+1}(z)$$

we obtain

$$1 - \frac{2\nu+1}{z} \frac{J_{\nu+1}(z)}{J_\nu(z)} + \left(\frac{J_{\nu+1}(z)}{J_\nu(z)}\right)^2 = \sum_{n=1}^{\infty} \zeta_\nu(2n)\, 2(2n-1) z^{2n-2}. \tag{2.5}$$

Replacing $J_{\nu+1}(z)/J_\nu(z)$ with the RHS of (2.4), we get a sum of series which can be recast into the form

$$1 + \sum_{n=1}^{\infty} \left[-4(\nu+n)\zeta_\nu(2n) + 4 \sum_{l=1}^{n-1} \zeta_\nu(2l)\zeta_\nu(2n-2l) \right] z^{2n-2} = 0. \tag{2.6}$$

Since it must be satisfied for any z with $|z| < j_{\nu 1}$, the identity must hold separately for every power of this variable. In particular, for $n = 1$ we are led to

$$\zeta_\nu(2) = \frac{1}{4(\nu+1)} \tag{2.7}$$

while, for $n \geqslant 2$, the 'quadratic' recursive relation

$$\zeta_\nu(2n) = \frac{1}{n+\nu} \sum_{l=1}^{n-1} \zeta_\nu(2l)\zeta_\nu(2n-2l) \tag{2.8}$$

follows. In the appendix we present an alternative derivation of this formula by direct integration over a convenient contour in the complex plane.

This equation, together with the previous, provides the value of $\zeta_\nu(2n)$ for any even integer argument. The first results thus obtained are:

$$\zeta_\nu(4) = \frac{1}{2^4(\nu+1)^2(\nu+2)} \tag{2.9}$$

$$\zeta_\nu(6) = \frac{1}{2^5(\nu+1)^3(\nu+2)(\nu+3)} \tag{2.10}$$

$$\zeta_\nu(8) = \frac{5\nu+11}{2^8(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)} \tag{2.11}$$

$$\zeta_\nu(10) = \frac{7\nu+19}{2^9(\nu+1)^5(\nu+2)^2(\nu+3)(\nu+4)(\nu+5)} \tag{2.12}$$

$$\vdots$$

These expressions are quoted by Watson [4] as due to Rayleigh. They are given without proof in his treatise and are generally overlooked. Now, with the help of (2.8), our derivation above has been straightforward (for a comparison, see the old derivation by Cayley [5]).

An immediate consequence of (2.8) is the non-trivial equality

$$\sum_{l=1}^{n-1} \sum_{\substack{k,m=1 \\ k\neq m}}^{\infty} \frac{1}{j_{\nu k}^{2l} j_{\nu m}^{2n-2l}} = (\nu+1)\zeta_\nu(2n). \tag{2.13}$$

The procedure described here has application to other functions that can be expressed as infinite products. For instance, we take $\sin z = z\prod_{n=1}^{\infty}(1 - z^2/\pi^2 n^2)$. Its logarithmic derivative gives (assuming $|z| < \pi$) a known expansion for ctg z

$$\operatorname{ctg} z = \frac{1}{z} - 2\sum_{m=0}^{\infty} \zeta(2m+2)\frac{z^{2m+1}}{\pi^{2m+2}} \tag{2.14}$$

where ζ is the Riemann zeta function itself. Differentiating with respect to z, the LHS will be $-\csc^2 z = -(1 + \operatorname{ctg}^2 z)$. Replacing ctg z with the series (2.14), and the RHS with its derivative, we obtain an identity which must be fulfilled for every power of z. Thus

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(2n) &= \frac{1}{n+\frac{1}{2}} \sum_{l=1}^{n-1} \zeta(2n-2l)\zeta(2l) \qquad n \geqslant 2. \end{aligned} \tag{2.15}$$

This provides a method for the recurrent evaluation of $\zeta(2n)$ from just the knowledge of $\zeta(2)$. Bearing in mind the relation between zeta function and Bernoulli numbers, $\zeta(2n) = (-1)^{n+1}[(2\pi)^{2n}/2(2n)!]B_{2n}$, the previous equations read

$$B_2 = \tfrac{1}{6}$$
$$B_{2n} = -\frac{1}{2n+1}\sum_{l=1}^{n-1}\binom{2n}{2l} B_{2n-2l}B_{2l} \qquad n \geqslant 2. \tag{2.16}$$

They supply a different way of finding all the non-vanishing Bernoulli numbers (B_1 aside) from the value of B_2. Equation (2.15) also yields

$$\sum_{l=1}^{n-1}\sum_{\substack{k,m=1\\ k\neq m}}^{\infty}\frac{1}{k^{2l}}\frac{1}{m^{2n-2l}} = \tfrac{3}{2}\zeta(2n). \tag{2.17}$$

Actually, all these results can also be obtained from those for ζ_ν by taking $\nu = \frac{1}{2}$, since $J_{1/2}(z) = \sqrt{2/\pi z}\sin z$.

2.2. Linear recursion

The above recurrence relation has turned out to be a powerful tool for successively obtaining the zeta functions of even argument. But, having succeeded in finding a 'second-order' law—as the RHS involves products of zeta functions at lower arguments—we would also like to have a linear rule available. The example of Euler's method (shown, e.g. in [4, p 500]) takes advantage of a *linear* recurrence among the $\zeta_0(2n)$'s. Now, our aim is to find the general form of such types of relation, thus extending the procedure to any ν.

We start, once more, by taking (2.4), this time written as

$$J_{\nu+1}(z) = 2J_\nu(z)\sum_{n=0}^{\infty}\zeta_\nu(2n+2)z^{2n+1}. \tag{2.18}$$

Next, replacing the J_ν's with their power series expansions

$$J_\nu(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+\nu)!}\left(\frac{z}{2}\right)^{2k+\nu} \tag{2.19}$$

(in general, these factorials are to be understood as gamma functions $(k+\nu)! \equiv \Gamma(k+\nu+1)$), we obtain a relationship between series which, after some index rearrangements, reads

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+\nu}}\left[\frac{1}{2\,n!(n+\nu+1)!} - 2\sum_{k=0}^{n}\frac{(-1)^k 2^{2k}}{(n-k)!(n-k+\nu)!}\zeta_\nu(2k+2)\right]z^{2n+\nu+1} = 0. \tag{2.20}$$

Validity for any z calls for the vanishing of every coefficient. Therefore

$$\frac{1}{4(n+\nu+1)} = \sum_{k=0}^{n}(-1)^k 4^k \frac{n!}{(n-k)!}\frac{(n+\nu)!}{(n+\nu-k)!}\zeta_\nu(2k+2) \qquad n \geqslant 0 \tag{2.21}$$

which amounts to

$$\frac{1}{4(\nu+n)}=\sum_{k=0}^{n-1}(-1)^k 4^k (k!)^2 \binom{n-1}{k}\binom{\nu+n-1}{k}\zeta_\nu(2k+2) \qquad n\geqslant 1. \tag{2.22}$$

This is the linear relation we were looking for.

The first resulting identities are

$$\frac{1}{4(\nu+1)}=\zeta_\nu(2) \tag{2.23}$$

$$\frac{1}{4(\nu+2)}=\zeta_\nu(2)-4(\nu+1)\zeta_\nu(4) \tag{2.24}$$

$$\frac{1}{4(\nu+3)}=\zeta_\nu(2)-8(\nu+2)\zeta_\nu(4)+32(\nu+1)(\nu+2)\zeta_\nu(6) \tag{2.25}$$

$$\frac{1}{4(\nu+4)}=\zeta_\nu(2)-12(\nu+3)\zeta_\nu(4)+96(\nu+2)(\nu+3)\zeta_\nu(6) -384(\nu+3)(\nu+2)(\nu+1)\zeta_\nu(8) \tag{2.26}$$

$$\frac{1}{4(\nu+5)}=\zeta_\nu(2)-16(\nu+4)\zeta_\nu(4)+192(\nu+4)(\nu+3)\zeta_\nu(6) -1536(\nu+4)(\nu+3)(\nu+2)\zeta_\nu(8) +6144(\nu+4)(\nu+3)(\nu+2)(\nu+1)\zeta_\nu(10). \tag{2.27}$$

This can be regarded as a system of linear equations in $\zeta_\nu(2), \zeta_\nu(4), \ldots, \zeta_\nu(10)$, which can be solved by repeated substitution from the first equation. It can be checked that its solution is (2.7), (2.9), (2.10), (2.11), (2.12). The following remark is in order: although a linear recursion looks in principle simpler than one involving products, in practice it is much easier to work with (2.8) than with (2.22).

3. Numerical evaluation of the ground state energies

Euler's method, combined with (2.7) and (2.8), turns out to be a very efficient procedure for the computation of $j_{\nu,1}$, associated with the ground state (here $\nu=|m-\alpha|+p+1/2$). The lower and upper bounds in (1.15) are now $[\zeta_\nu(2k)]^{-1/(2k)}$ and $\sqrt{\zeta_\nu(2k)/\zeta_\nu(2k+2)}$, respectively, where the square root comes from the factor 2 in the argument when going from ζ_ν to ζ_A. We calculate them for increasing k's until convergence of both the lower and the upper successions becomes apparent. At every step, a new $\zeta_\nu(2k)$ is found by means of a recursive function—implementing (2.7) and (2.8)—which is part of a simple C program. Table 1 shows the intermediate results obtained for the cases $\nu=0, 1/2$ and 1, whose first non-vanishing zeros are known to be 2.404826..., π and 3.831706..., respectively. The figures obtained after arriving at $k=10$ for different ν's between 0 and 1 are displayed in the table 2. Of these results, only those for $\nu\geqslant 1/2$ can be physically meaningful, as the

Table 1. Successive values in the approximation to $j_{\nu 1}$, corresponding to $\nu = 0, 1/2, 1$. For each ν, the values in the left and right columns are the lower and upper bounds, obtained as $[\zeta_\nu(2k)]^{-1/(2k)}$ and $[\zeta_\nu(2k)/\zeta_\nu(2k+2)]^{1/2}$ respectively.

k	$\nu = 0$		$\nu = 1/2$		$\nu = 1$	
1	2.000000	2.828427	2.449490	3.872983	2.828427	4.898979
2	2.378414	2.449490	3.080070	3.240370	3.722419	4.000000
3	2.401874	2.412091	3.132603	3.162278	3.812737	3.872983
4	2.404424	2.406133	3.139995	3.146427	3.827710	3.843076
5	2.404766	2.405069	3.141280	3.142768	3.830778	3.834980
6	2.404816	2.404871	3.141528	3.141883	3.831478	3.832667
7	2.404824	2.404834	3.141579	3.141665	3.831648	3.831990
8	2.404825	2.404827	3.141590	3.141611	3.831691	3.831791
9	2.404826	2.404826	3.141592	3.141597	3.831702	3.831731
10	2.404826	2.404826	3.141593	3.141594	3.831705	3.831713

Table 2. Different $j_{\nu 1}$'s for some values of ν between 0 and 1.

ν	$j_{\nu 1}$	ν	$j_{\nu 1}$
0	2.404826	0.6	3.282545
0.1	2.557451	0.7	3.421890
0.2	2.707073	0.8	3.559780
0.3	2.854097	0.9	3.696347
0.4	2.998849	1	3.831705
0.5	3.141593		

lowest possible ground state corresponds to $\nu = 1/2$. Such a value would be attained only for $\alpha = 0$, while higher ν's correspond to non-vanishing α's.

The calculation has been repeated—with the same accuracy—using a program which implements the linear recursion instead of the quadratic one. None of the results shown in the tables have changed, alhough the execution time is now shorter. However, away from this range the numerical errors produced by both algorithms may differ. Actually, for ν close to 100, we have observed differences of the orders of 10^{-6} and even 10^{-5} within the first ten steps.

4. Conclusions

We have focused in this paper on the derivation of exact and explicit formulas for the zeta function corresponding to the zeros of the Bessel function, $\zeta_\nu(2k)$ (for arbitrary ν), and on their systematic use for obtaining estimates of ground states of related Hamiltonian operators. The accuracy of the procedure developed here has proven to be excellent. Yet the main advantage of the method has not been exploited fully. By this we mean the possibility of finding such approximations *even in cases where the spectrum does not emerge from standard special functions*, but ζ_A can nevertheless be calculated, at least for some special values [6].

Another subject to be explored is the deformation of the spherical domain. The corresponding case in two dimensions was tackled by means of conformal mappings between the unit disc and other non-circular domains [2]. In three dimensions the difficulty would surely increase because such powerful complex transformations are lacking. We think that these subjects are worth studying in further detail.

Here in particular, by dealing with a problem of known spectrum, we have been able to relate the physical information concerning the energy levels with some remarkable numerical properties involving Bernoulli numbers and ordinary Riemann zeta functions.

Acknowledgments

This work has been partially supported by DGICYT (Spain), research project PB90-0022. SL acknowledges a grant from Generalitat de Catalunya.

Appendix. Derivation of the quadratic recurrence by complex integration

The zeta function

$$\zeta_\nu(z) \equiv \sum_{m=1}^{\infty} \lambda_m^{-z} \qquad J_\nu(\lambda_m) = 0 \tag{A.1}$$

can be written as the integral

$$\zeta_\nu(z) = -\frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}t\, \frac{J_\nu'(t)}{J_\nu(t)}\, t^{-z} = \frac{\mathrm{i}z}{2\pi} \int_C \mathrm{d}t\, \ln[J(t)]\, t^{-z-1} \tag{A.2}$$

where C is the sector-like contour of the complex plane domain $|\arg(t)| \leqslant \theta$, with $0 < \theta \leqslant \pi/2$, and $|t| \geqslant \epsilon$, with $0 < \epsilon < \lambda_1$ and $\mathrm{Re}\ t > 1$. Recalling that, for $|\arg(w)| < \pi - \delta$ ($\delta > 0$) and $|w| \to \infty$

$$J_\nu(w) \sim \sqrt{\frac{2}{\pi w}} \left[P_\nu(w) \cos\left(w - (\nu + \tfrac{1}{2})\frac{\pi}{2} \right) - Q_\nu(w) \sin\left(w - (\nu + \tfrac{1}{2})\frac{\pi}{2} \right) \right] \tag{A.3}$$

where P_ν and Q_ν are series of negative powers in w (we assume that $\nu > -1/2$), and particularizing it for the case $w = \rho\, \mathrm{e}^{\pm \mathrm{i}\pi/2}$ ($\rho > 0$), after substitution in expression (A.2), we obtain

$$\begin{aligned} \zeta_\nu(z) = \frac{\mathrm{i}z}{2\pi} \Bigg\{ \sum_{\pm} \int_{C_{\pm(\epsilon)}} \mathrm{d}t\, t^{-z-1} \ln\left[\sqrt{2\pi t}\, \exp\left(\pm \mathrm{i}\left(t - \frac{\pi}{2}(\nu + \tfrac{1}{2}) \right) \right) J(t) \right] \\ + 2\left[\frac{\mathrm{i}\epsilon^{-z+1}}{1-z} + \mathrm{i}(\nu + \tfrac{1}{2}) \frac{\pi}{2z} \epsilon^{-z} \right] \Bigg\} \end{aligned} \tag{A.4}$$

where we have taken $\theta = \pi/2$ and $C_\pm(\epsilon)$ are the two halves of the small circular part of the contour, of radius ϵ. The two integrals in (A.4) are analytical functions of z, while the last term is an explicitly meromorphic function. We thus see that ζ_ν has simple pole at $z = 1$ with residue $1/\pi$.

Let us now consider the restriction of ζ_ν to the strip $-1 < \mathrm{Re}\ z < 0$. The limit $\epsilon \to 0$ yields the fundamental expression

$$\zeta_\nu(z) = \frac{z}{\pi} \sin(\tfrac{1}{2}\pi z) \int_0^\infty \mathrm{d}\rho\, \rho^{-z-1} \ln\left[\sqrt{2\pi\rho}\, \mathrm{e}^{-\rho} I_\nu(\rho) \right] \qquad -1 < \mathrm{Re}\, z < 0. \tag{A.5}$$

It can be continued—for the value of Re z—to the right (and also to the left) of the real axis. Adding and subtracting to the integrand a term like $e^{-\rho}\sum_{k=0}^{2p}\rho^k(a_k+b_k\ln\rho)$, with $a_0\equiv\ln[\sqrt{2\pi}/(2^\nu\nu!)]$, $b_0\equiv\nu+1/2$, and adjusting the remaining a's and b's so that the difference from the integrand in (A.5) is

$$\ln\left[\sqrt{2\pi\rho}\,e^{-\rho}I_\nu(\rho)\right]-e^{-\rho}\sum_{k=0}^{2p}\rho^k(a_k+b_k\ln\rho)\simeq O(\rho^{2p+1}\ln\rho) \tag{A.6}$$

when $\rho\to 0$, it can be proven that the function $G_\nu(\rho)\equiv\ln\left[2^\nu\nu!\rho^{-\nu}e^{-\rho}I_\nu(\rho)\right]$ satisfies

$$G''+G'^2+G'\left(2+\frac{2\nu+1}{\rho}\right)+\frac{2\nu+1}{\rho}=0 \tag{A.7}$$

which has a solution of the form $G(\rho)=\sum_{k=1}^{\infty}c_k\rho^k$, $c_1=-1$, in a neighbourhood of $\rho=0$. In terms of $G_\nu(\rho)$ and $H_\nu(\rho)\equiv e^\rho G_\nu(\rho)$, the condition (A.6) is written as

$$H_\nu(\rho)-\sum_{k=1}^{2p}a_k\rho^k+(e^\rho-1)\ln\left(\frac{\sqrt{2\pi}}{2^\nu\nu!}\right)+M_\nu(\rho)\ln\rho\simeq O(\rho^{2p+1}\ln\rho) \tag{A.8}$$

where $M_\nu(\rho)$ is analytical around $\rho=0$ and contains all the information on the b coefficients—which are rigorously proven to exist but will not interest us here. The a's are read off from (A.8), with the result

$$a_k=\frac{1}{k!}\ln\left(\frac{\sqrt{2\pi}}{2^\nu\nu!}\right)+\sum_{j=1}^{k}\frac{c_j}{(k-j)!}. \tag{A.9}$$

Looking back to (A.5), we see that

$$\zeta_\nu(z)=\frac{z}{\pi}\sin(\tfrac{1}{2}\pi z)\int_0^\infty d\rho\,\rho^{-z-1}\left\{\ln\left[\sqrt{2\pi\rho}\,e^{-\rho}I_\nu(\rho)\right]-e^{-\rho}\sum_{k=0}^{2p}(a_k+b_k\ln\rho)\rho^k\right\}$$
$$+\frac{z}{\pi}\sin(\tfrac{1}{2}\pi z)\sum_{k=0}^{2p}[a_k\Gamma(k-z)+b_k\Gamma'(k-z)] \tag{A.10}$$

with the a's and b's determined as above, provides the desired analytical extension in the strip $-1<\mathrm{Re}\,z<2p+1$. After some straigthforward manipulations we see that for z a positive even integer, we get

$$\zeta_\nu(2m)=(-1)^{m+1}m\,c_{2m}. \tag{A.11}$$

It is also clear that the b's do not contribute to the expression in the limit and, by working out the precise value of the c's from (A.8), we obtain $\zeta_\nu(2)=[4(\nu+1)]^{-1}$ and the quadratic recurrence

$$\zeta_\nu(2(p+1))=\frac{1}{p+\nu+1}\sum_{k=0}^{p-1}\zeta_\nu(2(k+1))\zeta_\nu(2(p-k)) \tag{A.12}$$

valid for any integer p, $p\geqslant 1$.

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